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Abstract. Conformal mechanics, the simplest conformal field theory, is reformulated as a $d=1$ non-linear sigma model on the group SO(1, 2). Its action and equation of motion are shown to have a simple representation in terms of corresponding Cartan forms. The equation of motion amounts to certain algebraic relations between these forms which define a class of geodesics on SO(1, 2). Our geometric approach demonstrates deep analogies between the $d=1$ conformal mechanics and the $d=2$ Liouville theory. It is equally applicable to more complicated cases of superconformal mechanics and can be used to deduce the equations of the latter in a manifestly invariant superfield form.

1. Introduction

Conformally invariant field theories are of use in a wide range of phenomena. Conformal models in two dimensions are of particular interest as they constitute a field-theoretical basis of strings and superstrings. They describe possible string compactifications, provide explicit field realisations of Virasoro and super-Virasoro algebras and make it easy to establish a correspondence between the string theory and the $d=2$ statistical systems, etc (see, e.g., Goddard and Olive 1986). The geometric structure of these models is expected to encode the characteristic features of the geometry underlying string and superstring dynamics and so it deserves thorough analysis.

Many aspects of $d=2$ field theory are well modelled by its $d=1$ prototype, i.e. the quantum mechanics. In particular, the theories of a point particle and a superparticle have been intensively studied in recent years, with attention focusing on their similarities to the string and superstring theories. A good deal of attention has been paid to supersymmetric quantum mechanics which has interesting applications in its own right (see, e.g., Witten 1981). In view of the important role of conformal field theory it seems instructive to study conformal (De Alfaro *et al* 1974) and superconformal (Akulov and Pashnev 1983, Fubini and Rabinovici 1984) mechanics as these provide the simplest examples of such a theory. They reveal interesting analogies with the special class of $d=2$ conformal models, namely the Liouville and super-Liouville models. The latter have profound implications in string and superstring theories (see, e.g., Polyakov 1981a, b, Gervais and Neveu 1982) and exhibit remarkable geometric properties, such as full integrability.

In the pioneering paper by De Alfaro *et al* (1974) as well as in subsequent papers (Akulov and Pashnev 1983, Fubini and Rabinovici 1984) devoted to supersymmetric versions of conformal mechanics the main emphasis was on quantum mechanical aspects of these models (the spectrum, the structure of Hilbert space, etc). At the same time, their basic geometry was not understood in full generality even at the classical level. Such an understanding might be conducive both to achieving a deeper insight into the geometry of $d=2$ conformal theories (e.g. the Liouville and super-Liouville theories) and to constructing higher-N superextensions of conformal mechanics†. Up to now, only the $N=2$ and $N=4$ superconformal mechanics have been constructed (Akulov and Pashnev 1983, Fubini and Rabinovici 1984). A manifestly invariant superfield off-shell formulation was given only for the $N=2$ case (Akulov and Pashnev 1983).

In the present and forthcoming papers we propose a universal geometric framework for treating conformal mechanics and its superconformal extensions. These systems will be shown to be related to the geodesic motion on group manifolds of $d=1$ conformal and superconformal groups. The basis of our consideration is the covariant reduction method already developed by two of us (Ivanov and Krivonos 1983, 1984b) for application to $d=2$ Liouville-type systems. This proved to be an effective tool for algorithmic construction of higher-N superextensions of the Liouville equation and was recently used to set up a new wide class of $d=2$ superconformal sigma models with the Wess-Zumino action (Ivanov and Krivonos 1984a, c, d, Ivanov *et al* 1988). Geometrically, this method amounts to singling out certain finite-dimensional geodesic hypersurfaces in infinite-dimensional coset manifolds of $d=2$ conformal and superconformal groups. The Liouville and super-Liouville equations naturally emerge as the most essential conditions among those specifying these hypersurfaces. To put the method into use, one merely needs to know the structure relations of the corresponding $d=2$ superconformal algebra.

The equations of conformal and superconformal mechanics are generated when applying the same techniques to the group spaces of $d=1$ conformal and superconformal groups. These groups are finite-dimensional so everything is simpler than in the $d=2$ case. This makes it possible to understand more clearly the geometric meaning of covariant reduction.

In the present paper we give an account of our approach by the simplest example of bosonic ($N=0$) conformal mechanics. Our consideration will be purely classical. The supersymmetric case will be treated in a forthcoming paper where we will construct the off-shell superfield formulation of $N=4$ superconformal mechanics.

This paper is organised as follows. In § 2 we interpret conformal mechanics in terms of Cartan 1-forms on the parameter space of the $d=1$ conformal group SO(1, 2) subject to a kind of covariant reduction. In § 3 we explain the geometric meaning of this procedure and prove that the equation of conformal mechanics defines a class of geodesics on the group manifold. A simple geometric method of integrating this equation is also presented. It admits a straightforward extension to more complicated cases including the supersymmetric case.

In the appendix we establish the relationship with the customary description of geodesics in terms of the metric on the manifold.

2. Conformal mechanics and the non-linear realisation of the group SO(1, 2)

We begin by recalling the basics of conformal mechanics. It is defined by the equation (De Alfaro *et al* 1974) (we consider the one-component case):

$$\ddot{\rho}(t)=\gamma^2/\rho^3 \qquad [\gamma^2]=\mathrm{cm}^{-2} \qquad [\rho]=\mathrm{cm}^0 \tag{2.1}$$

† By N we mean the number of real spinor generators.

which follows from the action

$$S=\frac{1}{\lambda^2}\int \mathrm{d}t\left((\dot\rho)^2-\frac{\gamma^2}{\rho^2}\right) \qquad [\lambda^2]=\mathrm{cm}^{-1}. \tag{2.2}$$

The system (2.1) and (2.2) respects invariance under transformations of the $d=1$ conformal group SO(1, 2):

$$\begin{aligned}\delta t &= a+bt+ct^2=f(t) \\ \delta\rho(t) &= \tfrac{1}{2}\dot f(t)\rho(t)\end{aligned} \tag{2.3}$$

where a, b, c are, respectively, infinitesimal parameters of $d=1$ translation (L_{-1}), dilatation (L_0) and conformal boost (L_{+1}). The generators L_n form the algebra so(1, 2) $\sim$ sl(2, R):

$$\mathrm{i}[L_n, L_m]=(n-m)L_{n+m} \qquad n, m=-1, 0, 1. \tag{2.4}$$

(The simplest representation of L_n is via Pauli matrices, $L_{\pm1}=\frac{1}{2}(\tau^1\mp \mathrm{i}\tau^2)$, $L_0=\frac{1}{2}\mathrm{i}\tau^3$.) This notation demonstrates that SO(1, 2) is a finite-dimensional prototype of $d=2$ conformal (i.e. Virasoro) algebra (and enters into the latter as a maximal subalgebra).

Our aim is to relate the system (2.1) and (2.2) to the geometry of the group SO(1, 2). It will be convenient to choose the following parametrisation of this group:

$$g(x^1, x^2, x^3)=\exp(\mathrm{i}x^1L_{-1})\exp(\mathrm{i}x^2L_{+1})\exp(\mathrm{i}x^3L_0). \tag{2.5}$$

Non-linear SO(1, 2) transformations in the space of parameters $\{x^i\}$ are induced by left multiplications of group element (2.5):

$$g_0(a, b, c)g(x^1, x^2, x^3)=g(x^{1\prime}, x^{2\prime}, x^{3\prime}) \tag{2.6}$$

$$\begin{aligned}\delta x^1 &= a+bx^1+c(x^1)^2\equiv f(x^1) \\ \delta x^2 &= \tfrac{1}{2}f''(x^1)-f'(x^1)x^2 \\ \delta x^3 &= f'(x^1).\end{aligned} \tag{2.7}$$

It is evident that x^1 and $\exp(\frac{1}{2}x^3)$ transform just as the quantities t and ρ in (2.3). In what follows, this will allow us to identify both sets. The property that the line submanifold $\{x^1\}$ is closed under the action of SO(1, 2) is related to the fact that x^1 parametrises the left coset of SO(1, 2) over the subgroup with generators L_{+1}, L_0.

The local geometric properties of the group manifold $\{x^1\}$ are specified by the left-invariant Cartan 1-forms

$$g^{-1}\,\mathrm{d}g=\mathrm{i}\omega^nL_n \tag{2.8}$$

$$\begin{aligned}\omega^{-1} &= \exp(-x^3)\,\mathrm{d}x^1 \\ \omega^0 &= \mathrm{d}x^3-2x^2\,\mathrm{d}x^1 \\ \omega^{+1} &= \exp(x^3)[\mathrm{d}x^2+(x^2)^2\,\mathrm{d}x^1]\end{aligned} \tag{2.9}$$

which are none other than the SO(1, 2) covariant differentials of coordinates. The invariant line element $\mathrm{d}S^2$ is constructed from these forms. Representing L_n by Pauli matrices and choosing an appropriate normalisation, $\mathrm{d}S^2$ can be written as

$$\begin{aligned}\mathrm{d}S^2 &= -\mathrm{Tr}(g^{-1}\,\mathrm{d}g\,g^{-1}\,\mathrm{d}g)=2\omega^{-1}\omega^{+1}-\tfrac{1}{2}\omega^0\omega^0 \\ &= 2\mathrm{d}x^1\,\mathrm{d}x^2-\tfrac{1}{2}(\mathrm{d}x^3)^2+2x^2\,\mathrm{d}x^1\,\mathrm{d}x^3 \\ &\equiv g_{ij}\,\mathrm{d}x^i\,\mathrm{d}x^j.\end{aligned} \tag{2.10}$$

Let us now identify x^1 with the time t and consider an arbitrary curve in $\{x^i\}$,

$$t=x^1 \qquad x^2=x^2(t) \qquad x^3=x^3(t). \tag{2.11}$$

Now the group SO(1, 2) is parametrised by the time t and the Goldstone fields $x^2(t)$, $x^3(t)$ which specify the embedding of the curve in $\{x^i\}$ and correspond to the conformal boost and dilatation, respectively. Thus we are left with the non-linear realisation (Coleman *et al* 1969, Callan *et al* 1969, Volkov 1973, Ogievetsky 1974) of the $d=1$ conformal group. At this stage it is convenient to pass to the quantities with physical dimension $[(t]=\mathrm{cm}$, $[x^2]=\mathrm{cm}^{-1}$, $[x^3]=\mathrm{cm}^0)$, making use of the automorphism of the algebra (2.4) $L_{-1}\to fL_{-1}$, $L_{+1}\to f^{-1}L_{+1}$, $L_0\to L_0$ where f is an arbitrary constant (it can be dimensionful).

So far, our consideration has been purely kinematical; the t dependence of fields $x^2(t)$, $x^3(t)$ has been unrestricted. Just as in the case of non-linear realisations of the $d=2$ conformal group (Ivanov and Krivonos 1983, 1984b), the dynamics arises as a result of imposing the covariant reduction conditions on coordinates $\{t, x^2(t), x^3(t)\}$. In general this reduction proceeds as follows (Ivanov and Krivonos 1983, 1984b). All the Cartan forms, except for those belonging to some subalgebra of the initial algebra, are set equal to zero. In the $d=2$ case such a subalgebra was chosen to be either so(1, 2) or the algebra of the $d=2$ Poincaré group. For the corresponding dilaton field there appeared, respectively, either the Liouville equation or the free massless one. In the present case, we will perform the reduction to a subalgebra with the one generator

$$R_0=L_{-1}+m^2L_{+1}. \tag{2.12}$$

One may check that this generator corresponds to the compact so(2) subalgebra of so(1, 2). Thus we impose the constraints

$$g^{-1}\,\mathrm{d}g=g_R^{-1}\,\mathrm{d}g_R=\mathrm{i}\omega^{-1}R_0 \tag{2.13}$$

which amount to the set of Pfaff equations

$$\omega^0=0\Rightarrow\tfrac{1}{2}\dot{x}^3=x^2 \tag{2.14a}$$

$$\omega^1=m^2\omega^{-1}\Rightarrow\dot{x}^2+(x^2)^2=m^2\exp(-2x^3). \tag{2.14b}$$

The first one (2.14a) is kinematical; it covariantly expresses the Goldstone field x^2 as the derivative of the dilaton, thereby realising the inverse Higgs phenomenon (Ivanov and Ogievetsky 1975). Indeed, it follows from the transformation laws (2.7) that x^2 transforms just as $\frac{1}{2}\dot{x}^3$. On substitution of the expression for x^2 into (2.14b) the latter becomes

$$\ddot{x}^3+\tfrac{1}{2}(\dot{x}^3)^2=2m^2\exp(-2x^3) \tag{2.15}$$

which is easily recognised as the equation of conformal mechanics (2.1) after identifying $\rho(t)=\exp(\frac{1}{2}x^3(t))$ and $\gamma^2=m^2$.

Thus, we have derived equation (2.1) starting with the group space of SO(1, 2) where SO(1, 2) is realised by left shifts, and further constraining covariant differentials of coordinates by (2.14). The geometric meaning of this procedure will be clarified in § 3. Here we would like to note that one might choose a more general combination of SO(1, 2) generators than in (2.12):

$$\tilde{R}_0=L_{-1}+\tilde{m}^2L_{+1}+2\alpha L_0. \tag{2.16}$$

Then, instead of (2.15), one would have the more general set of equations

$$\omega^0=2\alpha\omega^{-1}\Rightarrow\tfrac{1}{2}\dot{x}^3=x^2+\alpha\exp(-x^3) \tag{2.14a'}$$

$$\omega^{+1}=\tilde{m}^2\omega^{-1}\Rightarrow\dot{x}^2+(x^2)^2=\tilde{m}^2\exp(-2x^3). \tag{2.14b'}$$

Substitution of (2.14a') into (2.14b') produces (2.15) with m^2 replaced by $\tilde{m}^2-\alpha^2$:

$$\ddot{x}^3+\tfrac{1}{2}(\dot{x}^3)^2=2(\tilde{m}^2-\alpha^2)\exp(-2x^3). \tag{2.17}$$

So, for $\alpha^2<\tilde{m}^2$ we have the standard conformal mechanics while for $\alpha^2>\tilde{m}^2$ we get the 'hyperbolic' version of (2.1) (De Alfaro *et al* 1974). For $\alpha^2=\tilde{m}^2$ the equation reduces to the free one. These three different situations correspond to three possible non-equivalent covariant reductions of the manifold $\{x^i\}$. Indeed, it is a simple exercise to check that the generators (2.16), with the parameters $\tilde{m}^2$ and α^2 varying within the above three domains, cannot be related to each other by any SO(1, 2) rotation and so belong to different orbits in the group space of SO(1, 2). Actually, the term in $\tilde{R}_0$ which contains L_0 can always be removed by a proper SO(1, 2) rotation:

$$R_0=L_{-1}+(\tilde{m}^2-\alpha^2)L_{+1}=\exp(-\mathrm{i}\alpha L_{+1})\tilde{R}_0\exp(\mathrm{i}\alpha L_{+1}). \tag{2.18}$$

This amounts to a constant right shift of $g(x^1, x^2, x^3)$ as

$$g(x^1,\tilde{x}^2,x^3)=g(x^1,x^2,x^3)\exp(\mathrm{i}\alpha L_{+1})\Rightarrow\tilde{x}^2=x^2+\alpha\exp(-x^3). \tag{2.19}$$

In terms of $\tilde{x}^2$, t and x^3, the set (2.14') has the same form as (2.14), with m^2 replaced by $\tilde{m}^2-\alpha^2$. Different types of covariant reduction are thus associated with three non-equivalent one-dimensional subalgebras of so(1, 2):

$$L_{-1}+m^2L_{+1}\qquad L_{-1}-m^2L_{+1}\qquad L_{-1}. \tag{2.20}$$

Recall that the first subalgebra is so(2) while the second one is so(1, 1). As will be shown in § 3, these three patterns correspond to three non-equivalent classes of geodesics on SO(1, 2).

To close this section, we present a simple invariant first-order action for the system (2.14) in terms of differential 1-forms (2.9):

$$S=-\frac{1}{\lambda^2}\int_e[\omega^{+1}+m^2\omega^{-1}]=-\frac{1}{\lambda^2}\int\mathrm{d}t\{\exp(x^3(t))[\dot{x}^2(t)+(x^2)^2]+m^2\exp(-x^3(t))\}. \tag{2.21}$$

Varying x^2 yields (2.14a). Inserting this constraint back into (2.14b) puts the latter into the standard second-order form (2.1) (with $\rho(t)=\exp(\frac{1}{2}x^3(t))$).

3. Geometric interpretation

Let us explain the geometric meaning of constraints (2.14). We will discuss the reduction to the so(2) subalgebra (2.12), keeping in mind the relation (2.18) and the fact that the generators of other possible reduction subalgebras (listed in (2.20)) follow from R_0 (2.12) either by substitution $m\to\mathrm{i}m$ or by putting $m^2=0$.

Differential forms (2.9), being covariant differentials of SO(1, 2) coordinates x^i, specify infinitesimal shifts of x^i along three independent directions in $\{x^i\}$. The constraints (2.14) restrict this motion to the shift along a curve generated by the right action of the Abelian subgroup with generator R_0. Indeed, solving (2.13) for $g_R(t, x^2(t), x^3(t))$, we find that the most general solution is

$$g_R(t,x^2,x^3)=g_0(c^1,c^2,c^3)\exp[\mathrm{i}\tau(t)(L_{-1}+m^2L_{+1})] \tag{3.1}$$

where c^i are integration constants and

$$d\tau = \exp(-x^3(t))\, dt. \tag{3.2}$$

It is easy to argue that (3.1) defines a geodesic on the manifold $\{x^i\}$. It is known (see, e.g., Gilmor 1974), that the geodesic motion on the coset or group manifolds is generated by the right action of the group on the coset elements. In the group space, any such element specifies a point whence some geodesic grows. The geodesic as a whole is restored by multiplying this fixed element from the right by an element of a certain Abelian subgroup having as the group parameter the natural parameter along the curve (the group is assumed to be taken in the exponential parametrisation). The choice of this subgroup fixes the tangent to the geodesic at the origin. Thus, the geodesic on a group space is completely defined by choosing an initial group element and some one-parameter subgroup acting on the former element from the right.

The formula (3.1) fits ideally into this general scheme. To prove that the SO(1, 2) element (3.1) defines a geodesic, we merely need to show the identity of τ with the natural parameter S. Inserting (2.14) into the definition (2.10) and taking account of (3.2) one gets

$$dS^2 = 2m^2\omega^{-1}\omega^{-1} = 2m^2 \exp(-2x^3)(dt)^2 = 2m^2(d\tau)^2 \Rightarrow ds/d\tau = \sqrt{2}m \tag{3.3}$$

i.e. τ actually coincides with s (up to a constant shift and rescaling).

Expression (3.1) provides the general solution to the constraints (2.14) and, hence, to the conformal mechanics equation (2.15) (or (2.1)) which is equivalent to the set (2.14). So we have shown that this equation describes a class of geodesics on the group SO(1, 2), with the coordinate $x^1 = t$ chosen as the parameter along the geodesic. For these geodesics $ds^2 > 0$, so they can be called 'time-like'. Two other types of geodesics on SO(1, 2), which are obtained by the reduction to two other subalgebras among those listed in (2.20), correspond, respectively, to $ds^2 < 0$ and $ds^2 = 0$. Thus, they are 'space-like' or 'light-like'. In the latter case (described by the free $m^2 = 0$ version of (2.15)) $|S|$ cannot serve as the evolution parameter, while τ or t still can. In the appendix we establish the explicit relation to a more familiar description of geodesics in terms of the metric g_{ij} introduced in (2.10).

The geometric approach allows us to render a transparent meaning to the procedure of integrating (2.1). It is reduced now to finding the explicit expressions for the original variables $\{x^i\}$ in terms of elements of the on-shell matrix (3.1). The constant factor g_0 entering into (3.1) actually involves only two independent integration constants which parametrise the coset SO(1, 2)/SO(2). The third constant can always be absorbed into a redefinition of τ. It is convenient to choose g_0 to be

$$g_0 = \exp(ic^1 L_{-1}) \exp(ic^3 L_0). \tag{3.4}$$

Substituting the expression for $g_R(t, x^2, x^3)$ (2.5) into (3.1) one finds

$$\begin{aligned} t &= c^1 + e^{c^3} m^{-1} \tan(m\tau) \\ x^2 &= \tfrac{1}{2}\exp(-c^3) m \sin(2m\tau) \\ x^3 &= c^3 - 2 \ln \cos(m\tau) \end{aligned} \tag{3.5}$$

which yields the explicit parametrisation of the geodesic in terms of proper time τ (or s). In accordance with the geometric interpretation of (3.1) given above, we have (use is made of (3.3)):

$$t(s=0)=c^1 \qquad \left.\frac{\mathrm{d}t}{\mathrm{d}s}\right|_{s=0}=\frac{1}{\sqrt{2}m}\exp(c^3)$$
$$x^2(s=0)=0 \qquad \left.\frac{\mathrm{d}x^2}{\mathrm{d}s}\right|_{s=0}=\frac{m}{\sqrt{2}}\exp(-c^3) \tag{3.6}$$
$$x^3(s=0)=c^3 \qquad \left.\frac{\mathrm{d}x^3}{\mathrm{d}s}\right|_{s=0}=0$$

whence it follows that the constants c^1, c^3 parametrise an initial point on the geodesic. We also see that, up to an unessential rescaling, the coupling constant m defines the components of the tangent vector to the geodesic at this point.

It is a simple exercise to extract from equations (3.5) the general solution of (2.1),

$$\rho(t)\equiv\exp(\tfrac{1}{2}x^3(t))=[A(1+(B/A)t)^2+A^{-1}\,m^2t^2]^{1/2} \tag{3.7}$$

where

$$A=\exp(c^3)+m^2(c^1)^2\exp(-c^3) \qquad B=-c^1m^2\exp(-c^3). \tag{3.8}$$

Any other form of the solution is reduced to (3.7) by a redefinition of integration constants.

One may check that the general solution (3.7) is invariant under the action of the SO(2) subgroup by

$$R=L_{-1}+2(B/A)L_0+A^{-2}(m^2+B^2)L_1 \tag{3.9}$$

$$\delta^*\rho(t)=\tfrac{1}{2}\dot{f}_R(t)\rho(t)-f_R(t)\dot{\rho}(t)=0 \tag{3.10}$$

$$f_R(t)=a[1+2(B/A)t+A^{-2}(m^2+B^2)t^2]. \tag{3.11}$$

Thus, there occurs the dynamical spontaneous breaking of SO(1, 2) to SO(2)$\propto R$ (De Alfaro *et al* 1974). This phenomenon has a simple interpretation in terms of geodesics. The generator R is related to R_0 (2.12) via the SO(1, 2) rotation by the element g_0 (3.4):

$$R=A^{-1}g_0[L_{-1}+m^2L_{+1}]g_0^{-1}.$$

Therefore, the left action of $\exp(\mathrm{i}aR)$ on $g_R(x^1,x^2,x^3)$ (3.1) merely results in the shift of proper time τ by an amount aA^{-1}, without affecting the shape of the geodesic, $x^i(\tau)\to x^i(\tau+aA^{-1})$. In other words, the left action of $\exp(\mathrm{i}aR)$ generates the shift along a given geodesic. The action of SO(1, 2)/SO(2) transformations changes the integration constants and so transforms one geodesic into another.

It is worthwhile to note that the integration of (2.1) can also be viewed as a reparametrisation of the group space of SO(1, 2). Indeed, let us choose from the beginning a different parametrisation of SO(1, 2):

$$g(x^1,x^2,x^3)=g(c^1,c^3,\tau)=\exp[\mathrm{i}c^1L_{-1}]\exp[\mathrm{i}c^3L_0]\exp[\mathrm{i}\tau(L_{-1}+m^2L_{+1})].$$

Then (3.5) give the relation between the two equivalent parametrisations of SO(1, 2). The Cartan forms in this new parametrisation are as follows:

$$\omega^{-1}=\exp(-c^3)\mathrm{d}c^1\cos(2m\tau)+\mathrm{d}c^3(1/m)\sin(2m\tau)+(1/m^2)\omega^{+1}$$
$$\omega^0=-m\exp(-c^3)\mathrm{d}c^1\sin(2m\tau)+\mathrm{d}c^3\cos(2m\tau)$$
$$\omega^{+1}=m^2\{\mathrm{d}\tau+\tfrac{1}{2}\exp(-c^3)\mathrm{d}c^1[1-\cos(2m\tau)]-(1/2m)\,\mathrm{d}c^3\sin(2m\tau)\}.$$

One is free to impose the constraints (2.14) on any parametrisation. It is easy to check that in terms of new variables these constraints are reduced to

$$\mathrm{d}c^1/\mathrm{d}\tau=\mathrm{d}c^3/\mathrm{d}\tau=0\Rightarrow c^1 \text{ and } c^3 \text{ are constants.} \tag{3.12}$$

Expressing c^1 and c^3 in terms of original variables, one obtains two first integrals of (2.1) (it is convenient to pass to the variables A and B given by (3.8))†

$$A(t)=(\rho-t\dot{\rho})^2+(m^2/\rho^2)t^2 \qquad B(t)=\dot{\rho}(\rho-\dot{\rho}t)-(m^2/\rho^2)t \qquad \dot{A}=\dot{B}=0. \tag{3.13}$$

Note that the variables $c^1(t)$, $c^3(t)$ and $\tau(t)$ are in a sense analogous to the action-angle variables of two-dimensional integrable systems.

We would like to mention that one further way of solving (2.1) is to reduce the latter to the harmonic oscillator equation. Introducing $\tilde{\rho}=\rho^{-1}$ and going to the proper time τ by means of (3.2) one may rewrite (2.1) as

$$\mathrm{d}^2\tilde{\rho}/\mathrm{d}\tau^2+m^2\tilde{\rho}=0.$$

Solving this equation and expressing τ in terms of t from the first-order equation (3.2) one arrives again at the expression (3.7).

4. Conclusions

In this paper we have demonstrated that the covariant reduction method proposed originally for a unified geometric description of Liouville-type systems in two dimensions (Ivanov and Krivonos 1983, 1984b), applies equally as well to $d=1$ systems, i.e. the models of particle mechanics. The foundations of the method can be clearly understood when looking at the $d=1$ case. The simple example we have analysed here in detail is mostly of an illustrative character, though it would perhaps be of some interest to see what implications this geometric picture has for the quantum case. The actual power of the covariant reduction approach will be demonstrated in our forthcoming paper where this technique is applied to the $N=4$, $d=1$ superconformal group $\mathrm{SU}(1,\frac{1}{2})$ to construct a manifestly invariant superfield formulation of $N=4$ superconformal mechanics.

It is worth mentioning that the model we have considered belongs to a wide class of completely integrable $d=1$ systems. The list of corresponding potentials can be found, e.g., in the review by Olshanetsky and Perelomov (1981). An interesting task is to reproduce in our approach the remaining potentials from this list (and, perhaps, to discover the unknown ones), starting with a non-linear realisation of an appropriate group and imposing the covariant reduction constraints on the relevant Cartan 1-forms. Also, it would be desirable to understand the relationship with the general method of integrating these systems which has been proposed by Olshanetsky and Perelomov. The method is based on relating the equation associated with a given integrable potential to the free (or geodesic) motion on a certain higher-dimensional auxiliary space. So it bears some formal resemblance to our method. We would like to emphasise once again that the main merit of our scheme should be seen in its algorithmic character. Once one chooses the group and the covariant reduction subgroup (the latter can be in general non-Abelian), then deducing the relevant mechanical system and finding out its general solution proceeds straightforwardly. The question to be answered is, of course, whether all the $d=1$ integrable systems can be obtained in this way.

† Note that the energy $H=(\dot{\rho})^2+m^2/\rho^2$ is expressed in terms of these quantities by the simple formula $H=A^{-1}[m^2+B^2]$.

Now let us dwell on analogies with the Liouville equation which is the simplest completely integrable $d = 2$ system. These analogies are far-reaching, despite the fact that in the Liouville case one deals with an infinite number of degrees of freedom. This last matter results in one starting with the infinite-dimensional $d = 2$ conformal group. Therefore there appear infinitely many Pfaff equations of the type (2.14*a*) (Ivanov and Krivonos 1983, 1984b). By these equations, the infinitely many fields parametrising a coset of the $d = 2$ conformal group are expressed via a single dilaton field. The latter is a direct analogue of the field $x^3(t)$. The Liouville equation arises analogously to (2.14*b*). The $d = 2$ counterpart of $d = 1$ reduction subalgebra so(2) is the subalgebra so(1, 2) of the $d = 2$ conformal algebra. Thus the Cartan form surviving the covariant reduction lives on that so(1, 2). As a consequence of covariant reduction constraints and of the original Maurer–Cartan equations, the remaining form satisfies the standard zero curvature condition that expresses the fact of complete integrability of the Liouville equation.

The zero curvature conditions have no analogue in the $d = 1$ case because of a lack of 2-forms in one dimension. However, as we have seen, the first-order covariant reduction constraints can still be implemented and these have a transparent geometric meaning. Thus it seems that the covariant reduction scheme may bear a deeper relation to the concept of integrability than the conventional approach based on the zero curvature representation. It would be of interest to extend this scheme to other integrable $d = 2$ systems, in particular to chiral field models. We conjecture that the latter models are associated with geodesic hypersurfaces in group manifolds of Kac-Moody groups (see, e.g., Dolan 1981).

As a final remark, we would like to stress that the results of this paper and of Ivanov and Krivonos (1983, 1984b) demonstrate a close relation between the $d = 2$ and $d = 1$ Liouville-type systems on the one hand and the intrinsic geometry of $d = 2$ and $d = 1$ conformal groups on the other. Perhaps this fact deserves special attention in view of the recent growth of interest in the geometry of coset spaces of the $d = 2$ conformal group in the context of string field theory (Bowick and Rajeev 1987, Bars and Yankielowicz 1987). A finite-dimensional toy model of the latter based on the group SO(1, 2) was recently considered by Jemenez and Sierra (1988).

It should be pointed out once more that the basic purpose of the present paper was to describe generalities of the $d = 1$ version of the covariant reduction method. Its potentialities and concrete applications for constructing new $d = 1$ models will be considered elsewhere.

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Appendix. Relation to the standard description of geodesics on SO(1, 2)

We start with the metric g_{ij} defined by equation (2.10)

$$g_{ij} = \begin{pmatrix} 0 & 1 & x^2 \\ 1 & 0 & 0 \\ x^2 & 0 & -\frac{1}{2} \end{pmatrix} \qquad g^{ij} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & -2(x^2)^2 & 2x^2 \\ 0 & 2x^2 & -2 \end{pmatrix}. \tag{A1}$$

The equation for geodesics corresponding to this metric is

$$\ddot{x}^i + \Gamma^i_{jk}\dot{x}^j\dot{x}^k = 0$$
$$x^i = x^i(s) \qquad \dot{x}^i = \mathrm{d}x^i/\mathrm{d}s \qquad |g_{ij}\dot{x}^i\dot{x}^j| = 1. \tag{A2}$$

Here Γ^i_{jk} are Christoffel coefficients calculated by the standard rules of Riemann geometry. In components, (A2) amounts to the set

$$\ddot{x}^1 - \dot{x}^1\dot{x}^3 = 0 \tag{A3a}$$
$$\ddot{x}^2 + 2x^2\dot{x}^2\dot{x}^1 + \dot{x}^2\dot{x}^3 + 2(x^2)^2\dot{x}^1\dot{x}^3 = 0 \tag{A3b}$$
$$\ddot{x}^3 - 2\dot{x}^2\dot{x}^1 - 2x^2\dot{x}^1\dot{x}^3 = 0. \tag{A3c}$$

Let us show that any solution of (2.14) also solves (A3). Making the change of variables $S \to x^1 = t$ in (A3) and using (3.3), it is easy to check that (A3a) is satisfied identically. The rest of equation (A3) is checked by using (2.14) repeatedly.

Conversely, one may obtain the set (2.14) as a result of partial integration of (A3). Specialising to the 'time-like' case $\mathrm{d}s^2 > 0$, one readily obtains

$$\mathrm{d}x^1/\mathrm{d}s = \beta_1 \exp(x^3)$$
$$x^2 = \tfrac{1}{2}\dot{x}^3 - \beta_2 \exp(-x^3) \tag{A.4}$$
$$\tfrac{1}{2}\ddot{x}^3 + \tfrac{1}{4}(\dot{x}^3)^2 = (1/2\beta_1^2)\exp(-2x^3)$$

where β_1 and β_2 are integration constants and the derivatives are taken with respect to $t = x^1$. Upon identifying $\beta_2 = \alpha$ and $m^2 = 1/2\beta_1^2$, these equations coincide with equation (2.14′) and (2.17) and so are equivalent to (2.14). It is worthwhile to emphasise that the coupling constant m^2 appears as an integration constant in this scheme.

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